

# On the edge-integrity of some graphs and their complements

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## *Abstract*

In this paper the authors study the edge-integrity of graphs. Edge-integrity is a very useful measure of the vulnerability of a network, in particular a communication network, to disruption through the deletion of edges. A number of problems are examined, including some Nordhaus–Gaddum type results. Honest graphs, i.e. those which have the maximum possible edge-integrity, are also investigated. A number of interesting open problems are also posed.

## **Introduction**

The ‘stability’ of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes there is a loss in its effectiveness. Normally new nodes or links are added so that the network is reconstructed in an attempt to regain its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

Since such a network can be represented by a graph,  $G$ , with a vertex set,  $V(G)$ , and an edge set,  $E(G)$ , many graph theoretical parameters have been used in the past to describe the stability of communication networks. Most notably, the parameters called connectivity and edge-connectivity have been frequently used. The connectivity

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of a graph  $G$  is the least number of vertices of  $G$  whose removal disconnects  $G$ . Similarly, the edge-connectivity of  $G$  is the least number of edges whose removal disconnects  $G$ . The higher the connectivity (edge-connectivity) of  $G$  the more stable it is considered to be. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. That is, two graphs with the same number of vertices and the same connectivity may result in entirely different forms after a minimum disconnecting set of vertices is removed. One may be totally disconnected while the other may consist of a few very stable components, and thus be much easier to reconstruct. Consequently, a number of other parameters have recently been introduced in an attempt to cope with this difficulty. The parameters considered in this paper were introduced in [7] in order to deal with this problem. The (vertex-)integrity of a graph  $G$ ,  $I(G)$ , is defined as  $I(G) = \min \{|S| + m(G-S)\}$ , where the minimum is taken over all possible subsets  $S$  of  $V(G)$  and  $m(G-S)$  is the order of a largest component of  $G-S$ . The edge-integrity of  $G$ ,  $I'(G)$ , is similarly defined as  $I'(G) = \min \{|R| + m(G-R)\}$ , where the minimum is taken over all subsets  $R$  of  $E(G)$ . Thus a large integrity (edge-integrity) indicates that either a large number of vertices (edges) must be deleted or else at least one large component remains after the deletion of the vertices (edges).

In this paper the authors are concerned primarily with edge-integrity. We are interested in Nordhaus-Gaddum type results and the problem of which graphs are 'honest' with respect to edge-integrity. A graph  $G$  is said to be honest if  $I'(G) = p$ , the number of vertices in  $G$ . Note that in an honest graph the minimum is achieved for edge-integrity when no edges are deleted. Also note that the only graphs with  $I(G) = p$  are the complete graphs.

Throughout this paper all graphs are assumed to have no loops or multiple edges. The notation will be as in [9].

## Basic results

The parameters of integrity and edge-integrity were introduced by Barefoot, Entringer and Swart in [8] and were studied more extensively by the same authors in [7]. Computational aspects of these parameters were studied in [10, 11]. A number of results on integrity have been given by Bagga, Beineke, Lipman, and Pippert in [1-3, 5] and some results on edge-integrity have been given by the same authors in [4, 6]. Others have studied the integrity of certain classes of graphs in [13, 14].

Most of the following results were obtained by Barefoot, Entringer, and Swart in [7] and by Bagga, Beineke, Lipman and Pippert in [4].

**Proposition 1** ([4]). *If  $H$  is a subgraph of  $G$ , then  $I'(H) \leq I'(G)$ .*

**Proposition 2** ([7]). *Let  $T$  be a tree of order  $p$  and  $P_p$  the path of order  $p$  then*

$$\lceil 2\sqrt{p} \rceil - 1 = I'(P_p) \leq I'(T).$$

**Proposition 3.**  $I'(K_p) = p$ .

**Proposition 4.**  $I'(K_{1,n}) = n + 1$ .

**Theorem 1** ([4]). For any graph  $G$  and  $H$ ,

$$I'(G \times H) \leq \min \{ |V(G)| \cdot I'(H), |V(H)| \cdot I'(G) \}.$$

**Theorem 2** ([4]). For any graph  $G$ ,  $I'(K_n \times G) = nI'(G)$ .

An immediate consequence of this theorem is the following.

**Corollary 1** ([4]).  $I'(Q_n) = 2^n$ .

**Proposition 5** ([7]). For any graph  $G$ ,  $I'(G) \geq \Delta(G) + 1$ .

As can be seen from some of the above theorems many graphs are honest. In the study of edge-integrity the question of which graphs are honest is an interesting one. The following result will be very important in this study. In fact, Propositions 7 and 8 and Theorems 5 and 8 are all direct consequences of this result.

**Theorem 3** ([4]). For a graph  $G$  of order  $p$ , if  $\text{diam}(G) = 2$  then  $I'(G) = p$ .

**Corollary 2** ([4]). For a graph  $G$  of order  $p$ , if  $\deg(x) + \deg(y) \geq p - 1$  for every pair of non-adjacent vertices in  $G$  then  $I'(G) = p$ .

For any integer  $k$ ,  $3 \leq k < p$ , it is easy to find graphs with diameter 3 and edge-integrity  $k$ . Thus, in some sense Theorem 3 is the best possible.

### Nordhaus–Gaddum type results

The Nordhaus–Gaddum theorem is stated in [9] as follows.

**Theorem 4.** If  $G$  is a graph of order  $p$ , then:

- (a)  $2\sqrt{p} \leq \chi(G) + \chi(\bar{G}) \leq p + 1$ , and
- (b)  $p \leq \chi(G) \cdot \chi(\bar{G}) \leq ((p + 1)/2)^2$ .

Results of this type are called Nordhaus–Gaddum type results. These type results are the first that we shall consider. Hence we must consider the edge-integrity of the complements of some graphs. We will begin with some very simple results.

**Proposition 6.**  $I'(K_p) + I'(\bar{K}_p) = p + 1$ .

**Proposition 7.** For  $p \geq 5$ ,  $I'(P_p) + I'(\overline{P_p}) = p + \lceil 2\sqrt{p} \rceil - 1$ .

**Proof.** It is easy to see that  $\text{diam}(\overline{P_p}) = 2$  and so by Theorem 3,  $\overline{P_p}$  is honest and the result follows from Proposition 2.  $\square$

**Proposition 8.** If  $m \leq n$  then  $I'(K_{m,n}) + I'(\overline{K_{m,n}}) = m + 2n$ .

**Proof.** Again it is easy to see that  $\text{diam}(K_{m,n}) = 2$  and hence, by Theorem 3,  $K_{m,n}$  is honest. Note also that  $I'(\overline{K_{m,n}}) = n$  and hence the result.

**Theorem 5.** If  $G$  is  $r$ -regular with  $p$  vertices and  $r \geq (p-1)/2$  then  $I'(G) = p$ .

**Proof.** Clearly,  $\deg(x) + \deg(y) \geq p-1$  for all vertices  $x$  and  $y$  and so, by Corollary 2,  $I'(G) = p$ .  $\square$

**Corollary 3.** If  $G$  is  $r$ -regular with  $p$  vertices then either  $I'(G) = p$  or  $I'(\overline{G}) = p$ .

It is easy to see from the previous result that if  $G$  is regular then  $I'(G) + I'(\overline{G}) \geq p+1$ . This result is generalized in the following theorem.

**Theorem 6.** For any graph  $G$  with  $p$  vertices,  $I'(G) + I'(\overline{G}) \geq p+1$ , with equality iff  $G = K_p$  or  $G = \overline{K_p}$ .

**Proof.** From Proposition 5 we know that  $I'(G) \geq \Delta(G) + 1$ . But  $\delta(\overline{G}) = p - \Delta(G) - 1$  and so  $I'(\overline{G}) \geq \Delta(\overline{G}) + 1 \geq \delta(\overline{G}) + 1 = p - \Delta(G)$ . Hence  $I'(G) + I'(\overline{G}) \geq p+1$ .

If  $G$  is not regular then  $\Delta(\overline{G}) > \delta(\overline{G})$  and the above inequality is strict. Thus, if  $I'(G) + I'(\overline{G}) = p+1$  then  $G$  is regular. Suppose  $G$  is neither  $K_p$  nor  $\overline{K_p}$ . Then  $\overline{G}$  is neither  $K_p$  nor  $\overline{K_p}$ . So  $I'(G) > 1$  and  $I'(\overline{G}) > 1$ . But, by Corollary 3, either  $G$  or  $\overline{G}$  is honest, and so  $I'(G) + I'(\overline{G}) > p+1$ . Hence if  $I'(G) + I'(\overline{G}) = p+1$  then  $G = K_p$  or  $G = \overline{K_p}$ . From Proposition 6 we know that if  $G = K_p$  or  $G = \overline{K_p}$  then  $I'(G) + I'(\overline{G}) = p+1$  and this concludes the proof.  $\square$

**Corollary 4.** If  $G$  is self-complementary then  $I'(G) > (p+1)/2$ .

**Corollary 5.** If  $G$  is regular and self-complementary then  $G$  is honest.

**Theorem 7.** For  $n \geq 3$ , both  $Q_n$  and  $\overline{Q_n}$  are honest.

**Proof.** The graph  $Q_n$  is  $n$ -regular with  $2^n$ -vertices and so the graph  $\overline{Q_n}$  is  $(2^n - n - 1)$ -regular. But  $2^n - n - 1 \geq (2^n - 1)/2$ , and thus, by Theorem 5,  $\overline{Q_n}$  is honest and hence the result.  $\square$

This leads us to the following question.

**Question 1.** When is  $I'(G) + I'(\bar{G}) = 2p$ ?

In other words, when are both  $G$  and  $\bar{G}$  honest? We can see that there are a number of interesting graphs with this property, such as  $Q_n$  and  $C_5$ . From Theorem 3 we get the following corollary.

**Corollary 6.** *If  $\text{diam}(G) = \text{diam}(\bar{G}) = 2$  then both  $G$  and  $\bar{G}$  are honest.*

So we get another question, which is a weaker form of the previous question.

**Question 2.** For which graphs  $G$  is  $\text{diam}(G) = \text{diam}(\bar{G}) = 2$ ?

This seems to be a very interesting question, in light of the fact that diameter 2 graphs have received a great deal of attention. We will consider some such graphs in the next section.

## Graphs of diameter 2

In this section we will attempt to discover some honest graphs, and some graphs where both the graph itself and its complement are honest. At first thought it would seem that graphs with no induced  $P_4$  would have the property that both the graph and its complement have diameter 2, but this is not the case. It is true that if  $G$  has no induced  $P_4$  then  $\text{diam}(G) = 2$  and also  $\bar{G}$  has no induced  $P_4$ , but in this case  $\bar{G}$  must be disconnected [12]. There do exist graphs with the diameter of  $G$  and  $\bar{G}$  both equal to 2. For example,  $C_5$  has this property. We will now find two classes of graphs with this property. The circulant graph  $G = C(p; a_1, a_2, \dots, a_k)$  is defined as follows:  $V(G) = \{0, 1, \dots, p-1\}$  and  $E(G) = \{xy \mid x - y \equiv a_i \pmod{p}, 1 \leq i \leq k\}$ . The circulant graph  $G = C(3k+2; 1, 4, 7, \dots, 3k+1)$  has the property that  $\text{diam}(G) = \text{diam}(\bar{G}) = 2$ . To see this consider vertices  $i$  and  $j$  in  $V(G)$ ,  $ij \notin E(G)$ . Without loss of generality let  $i$  be 1. If  $j = 3t$ , for some  $t$  then  $(3t-1)j$  and  $1(3t-1)$  are elements of  $E(G)$ , so the distance between 1 and  $j$  is 2. Similarly, if  $j = 3t+1$  then  $j(3(t+1)-1)$  and  $1(3(t+1)-1)$  are both in  $E(G)$ . Thus,  $\text{diam}(G) = 2$ . Now consider  $j$  in  $V(G)$  such that  $1j \in E(G)$ . Then  $j = 3t-1$  for some  $t$  and so  $j(3t+1) \notin E(G)$  and  $1(3t+1) \notin E(G)$ . Hence,  $\text{diam}(\bar{G}) = 2$ . The graph  $C(8; 1, 4, 7)$  is seen drawn two different ways in Figs. 1(a) and 1(b). The drawing in Fig. 1(b) suggests another class of such graphs.

Define the graph  $G$  as follows. Let  $S = \{s_1, s_2, \dots, s_k\}$ , for  $k \geq 2$ , and also let  $T = \{t_1, t_2, \dots, t_m\}$ . Partition the set  $T$  into  $k$  subsets  $S_1, \dots, S_k$ , where each  $S_i$  is nonempty. Define  $V(G) = \{v\} \cup S \cup T$  and  $E(G) = \{vs_i \mid 1 \leq i \leq k\} \cup \{s_i t_j \mid t_j \in S_i, 1 \leq i \leq k\} \cup \{t_i t_j \mid t_i \in S_x, t_j \in S_y, x \neq y\}$ .  $G$  is drawn in Fig. 2. Notice that the subgraph induced by  $T$  is isomorphic to  $K_{|S_1|, |S_2|, \dots, |S_k|}$  and the subgraph induced by  $S$  is  $\bar{K}_k$ . It is easy to see that  $d(v, s_i) = 1$ ,  $d(v, t_i) = 2$ ,  $d(s_i, s_j) = 2$ ,  $d(s_i, t_j) = 1$  for  $t_j \in S_i$ ,  $d(s_i, t_j) = 2$

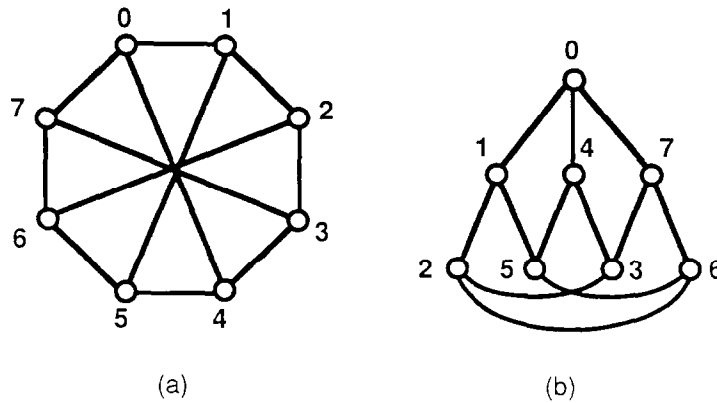


Fig. 1.

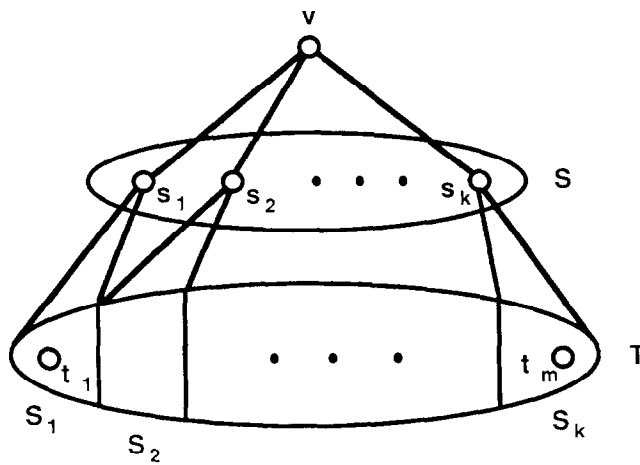


Fig. 2.

for  $t_j \notin S_i$ ,  $d(t_i, t_j) = 1$  for  $t_i \in S_x$  and  $t_j \notin S_x$ , and  $d(t_i, t_j) = 2$  for  $t_i, t_j \in S_x$ . Thus  $\text{diam}(G) = 2$ . It is also easy to see that for every edge  $ij$  in  $E(G)$ , there is a vertex that is not adjacent to either  $i$  or  $j$ . Hence  $\text{diam}(\bar{G}) = 2$ . Notice that adding an edge of the form  $t_i t_j$ , for  $t_i, t_j \in S_x$ , will not affect the property that  $\text{diam}(G) = \text{diam}(\bar{G}) = 2$ . Thus we now have two infinite classes of graphs with the property that  $I'(G) = I'(\bar{G}) = p$ . It is relatively easy to construct other such classes.

We shall now consider some graphs which always have  $I'(G) = p$ .

**Theorem 8.** For all graphs  $G$  and  $H$ , with  $|V(G)|, |V(H)| \geq 3$ ,  $\overline{G \times H}$  is honest.

**Proof.** Let  $G$  and  $H$  be graphs, each having at least three vertices and let  $(x, u)$  and  $(x, v)$  be adjacent vertices in  $G \times H$ . Then there exists a vertex  $y \neq x$  in  $V(G)$  and  $w \neq u, v$  in  $V(H)$ , since  $|V(H)| \geq 3$ . Thus  $(y, w)$  is a vertex in  $G \times H$  that is not adjacent to either  $(x, u)$  or  $(x, v)$  and so  $(y, w)$  is adjacent to both  $(x, u)$  and  $(x, v)$  in  $\overline{G \times H}$ . Hence  $\text{diam}(\overline{G \times H}) \leq 2$  and so, by Theorem 3,  $\overline{G \times H}$  is honest.

We will now give one final theorem.

**Theorem 9.** For any graph  $G$ ,  $I'(\overline{K_n} \times G) = \min \{n|R| + m(G - R)\}$ , where the minimum is taken over all subsets  $R$  of  $E(G)$ , i.e., one way to obtain the edge-integrity of  $\overline{K_n} \times G$  is by deleting the same set of edge from each copy of  $G$ .

**Proof.** Let  $H = \overline{K_n} \times G$ . Note that  $H$  is just  $n$  disjoint copies of  $G$ . Let  $G_i$  be the copies of  $G$ ,  $1 \leq i \leq n$ ,  $S$  be a set of edges of  $H$  such that  $|S| = \min \{|R| + m(H - R) \mid I'(H) = |R| + m(H - R)\}$  and  $S_i = S \cap E(G_i)$ . If  $S_i = \emptyset$  for some  $i$  then  $m(H) = m(G)$  and so  $S = \emptyset$  and  $S_i = \emptyset$  for  $1 \leq i \leq n$ . Hence  $I'(H) = m(G)$ . So assume  $S_i \neq \emptyset$  for  $1 \leq i \leq n$ . We will use the notation  $S_i = S_j$  to mean that in the isomorphic copies of  $G$ ,  $S_i$  will map to  $S_j$  under some isomorphism. If  $S_i \neq S_j$  for some  $i$  and  $j$  then we have 3 cases.

Case 1: If  $|S_i| = |S_j|$  and  $m(G_i - S_i) = m(G_j - S_j)$  then let  $S'_i = S_j$  and  $S' = (S - S_i) \cup S'_i$ .

Thus we have a set  $S'$  with  $I'(H) = |S'| + m(H - S')$  and  $S'$  restricted to  $G_i$  is equal to  $S'$  restricted to  $G_j$ .

Case 2: If  $|S_i| = |S_j|$  but  $m(G_i - S_i) > m(G_j - S_j)$ .

Again let  $S'_i = S_j$  and  $S' = (S - S_i) \cup S'_i$ . So  $|S| + m(H - S) \geq |S'| + m(H - S')$  and thus  $S'$  is a set with  $I'(H) = |S'| + m(H - S')$  and  $S'$  restricted to  $G_i$  is equal to  $S'$  restricted to  $G_j$ .

Case 3: If  $|S_i| > |S_j|$  then let  $S'_i = S_j$  and  $S' = (S - S_i) \cup S'_i$ .

Then  $|S'| < |S|$  and, since  $m(G_i - S'_i) = m(G_j - S_j)$ , we have  $m(H - S') \leq m(H - S)$ . Hence  $|S'| + m(H - S') < |S| + m(H - S)$ , a contradiction.

Thus there exists a set  $S$  with  $I'(H) = |S| + m(G - S)$  where  $S_i = S_j$  for all  $1 \leq i, j \leq n$ . Therefore  $I'(H) = n|S_1| + m(G - S_1) = \min \{n|R| + m(G - R)\}$ , where the minimum is taken over all subsets  $R$  of  $E(G)$ .  $\square$

Notice that what this theorem says is that if the edges of a graph  $G$  are all given a weight of  $n$ , then the 'weighted edge-integrity' of this graph (using the sum of the weights of the edges rather than the number of edges when computing the minimum) is the same as taking the edge-integrity of  $n$  copies of  $G$  without weights.

## Questions

These results lead to a number of open questions, two of which have already been stated.

**Question 3.** What are  $I'(C_n \times C_m)$  and  $I'(C(p; a_1, \dots, a_k))$ ?

**Question 4.** For which graphs  $G$ , besides  $P_4$ , are both  $G$  and  $\bar{G}$  not honest?

**Question 5.** If  $G$  is self-complementary, what is  $I'(G)$ ?

Also, in a manner similar to Theorem 8, what occurs if the vertices of  $G$  are now all assigned a weight of  $n$  and instead of taking the largest component of  $G - R$  we take the sum of the weights of the vertices in  $G - R$ ? This gives the following question.

**Question 6.** What is  $\min\{|R| + nm(G - R)\}$ , where the minimum is taken over all subsets  $R$  of  $E(G)$ ?

**Authors note.** We would like to point out that Question 4 has been answered by Bagga, Beineke, Lipman and Pippert in [6], which also appears in this volume, as follows.

**Theorem 10.** *If  $G \neq P_4$ , then either  $G$  or  $\bar{G}$  is honest.*

Note that this result subsumes a several results stated in this paper. In particular Corollaries 3, 4 and 5 are now unnecessary.

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